

# Indistinguishability as non-locality constraint

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## Abstract

Quantum mechanics has long bewildered many people and questionings about its consistency and completeness have been raised, such as the famous case of the Einstein-Podolsky-Rosen paradox. Nonetheless, quantum theory has established firm grounds for our understanding about microscopic phenomena, and non-locality and entanglement is nowadays considered an important resource for quantum information processing. However, it has been noticed that relativistic causality and non-locality alone, assumed as axioms, are not enough to explain the limit of two-qubit quantum correlations, known as Tsirelson's bound. In this paper, to obtain such explanation, indistinguishability is added as a fundamental principle to these two axioms set, standing as the source of interaction and correlation. Taken together, the three principles — no-signaling, non-locality, and indistinguishability — can reproduce Tsirelson's bound and offer a simple and elegant explanation to non-local quantum correlations.

Quantum theory has brought many new perspectives to how we see the world and how we understand nature<sup>1</sup>. The challenging notion of non-locality bewildered many scientists and was addressed by prominent physicists<sup>2,3</sup>. No-signaling, the prohibition of any information transfer faster than light, and non-locality can be seen as fundamental physical principles, but they alone allow for non-local correlations beyond that allowed in quantum mechanics, known as Tsirelson’s bound<sup>4,5</sup>. Non-locality’s counter-intuitive character has been called a “spooky action at distance” by Einstein, and Bell proved the incompatibility of local hidden variables with quantum theory<sup>3</sup>, reinforcing non-locality’s role. The same result was also independently obtained by Clauser et al.<sup>6</sup>; In their work, they show the violation, under a quantum mechanical context, of the inequality now known as Clauser-Horne-Shimony-Holt (CHSH) inequality, i.e.

$$|E(A_1, B_1) + E(A_2, B_1) + E(A_1, B_2) - E(A_2, B_2)| < 2, \quad (1)$$

that is valid for classical (hidden variable) theory. Here,  $E(A_i, B_j)$  gives the correlation of a bipartite system divided into  $A$  and  $B$ , on two different bases (entanglement witnesses) labeled with the indices  $i, j$ , with possible measurement values given by  $\pm 1$ . The violation of this inequality gives us a measure on how quantum correlation may go beyond classical (local) correlations in a bipartite system. Tsirelson showed the upper bound for CHSH inequality in a quantum theory is  $2\sqrt{2}$ , which comes from the structure of the Hilbert space<sup>4</sup>. CHSH inequality violation was obtained in careful recent experiments, asserting non-locality of quantum theory<sup>7-9</sup>. Nevertheless, Popescu and Rohrlich exposed the fact that only relativistic causality (no-signaling) and non-locality as axioms are not enough to physically justify Tsirelson’s bound<sup>5</sup>. Since then, explanations have arisen, based on information causality<sup>10</sup> (a generalization of relativistic causality), macroscopic locality<sup>11</sup>, and exclusivity principle<sup>12</sup>. In this work, one extra and relatively intuitive fundamental physical principle is proposed in order to close non-locality within (quantum) bounds: indistinguishability. By relating indistinguishability with symmetries, and deeming quantum interactions and correlations as originating from indistinguishability, it is shown that Tsirelson’s bound is the limit to correlations allowed in this framework. With this essential principle, it is implied that no-signaling, non-locality, and indistinguishability may elegantly close a three-set of fundamental physical principles.

Distinguishability is an important property in quantum theory, and has been addressed

in different contexts<sup>1,13–17</sup>. The present discussion shall be centered around basis-state indistinguishability, i.e. whether two or more measurement bases can be distinguished at a given region of spacetime. Here, basis indistinguishability is defined by expanding the definition used by Kawakubo and Koike<sup>17</sup> based on unambiguous distinguishability<sup>13–15</sup>.

**Definition 1 (basis indistinguishability)** *Given countable many bases in a set  $I$ , if there is no POVM  $\mathcal{O} = (\mathcal{O}_j)_{j \in J}$ , with  $J = I \cup \{u\}$  a countable set with one extra element than  $I$  for unidentifiable basis, so that  $\text{Tr}(\pi_i \mathcal{O}_j) = q_i \delta_{ij}$ , where  $\pi_i$  is a projection operator onto a basis  $i \in I$  and  $q_i$  a non-zero real value, bases in  $I$  are said perfectly **indistinguishable**. In this case,  $\text{Tr}(\pi_i \mathcal{O}_u) = 1 \forall i$ .*

Formally, particles can also be (effectively) *distinguished* by not fitting under the following condition (see, for instance, Eckert et al.<sup>16</sup>).

**Definition 2 (particle indistinguishability)** *Let  $\Psi(r_i)$  be a state composed by  $n$  (identical) particles, with density operator  $\rho_A$ , where  $r_i$  denotes the relevant coordinates that identify the  $i$ -th particle. The particles are said (partially) **indistinguishable** as long as a physically meaningful exchange correlation or exchange energy of two different particles at  $r_i, r_j$  exists.*

Definition 1 shall lay the foundation of what will be called **basis indistinguishability** throughout this paper, whilst definition 2 shall stand for distinguishing physical fractions of a system or for discussing **particle indistinguishability**. Note that exchanging particles is equivalent to exchanging bases, whence the two concepts become intertwined and equivalent, therefore we shall use simply *indistinguishability* in the present discussion, which is mainly concerned with basis indistinguishability.

The main concept assumed for the present analysis is that *interactions in quantum mechanics are a fundamental result of indistinguishability*. To understand it, consider all physical interactions to be fundamentally local and relativistic causal, a standard physical assumption. Therefore, particles far apart from each other (beyond appreciable long range interaction) are essentially out of bounds of mutual influence. It becomes sufficient condition for their exchange energy or exchange correlation to vanish, leaving the particles distinguishable, following definition 2. Nevertheless, it is hard to prove the existence of a necessary condition for our assumption. We are bound to have indistinguishability only when we have interaction, and this begs the question about whether it is possible to completely overcome

indistinguishability (i.e., to have perfect distinguishability) on an interacting system, since one could have particles relatively close but prevent interaction with some kind of barrier or shield. For a two-body interaction, there is no way to circumvent exchange correlation when interaction is present, but more complex many-body effects could create such a situation. As an example, one may think of frustrated systems where renormalization may weaken exchange integrals, reducing exchange energy. Nevertheless, even then some order tends to still be present, originated from the non-vanishing exchange energies and some finite degree of indistinguishability. Only a completely disordered “magnet” could be said fundamentally non-interacting, with all possible configurations degenerate as ground states effectively losing exchange interaction and rendering each component perfectly distinguishable from the others. This practical inviability to have interaction while perfectly distinguishing the interacting particles leaves us with no special reason to say that interaction is more fundamental than indistinguishability, therefore we take the freedom to choose indistinguishability as a more fundamental matter. Thus, we define interaction as

**Definition 3 Interaction:** *a physical measure of indistinguishability. When two or more particles are close enough within a region of spacetime, limiting the possibility of unambiguously distinguishing them, interaction happens.*<sup>18</sup>

It is also reasonable that indistinguishable particles (due to scattering or any other process) remain so until some other distinguish process takes place. Considering that quantum particles are generally divided into fermions and bosons<sup>19</sup>, the wavefunction for such particles must always be (anti-)symmetrized to account for the effects of exchange of identical and indistinguishable particles. Nevertheless, if exchange energies vanish and particles become essentially distinguishable, symmetrization can be safely disregarded. Once interactions take place, this is no longer the case and at least symmetrization between interacting particles must be taken, leading to a certain level of indistinguishability, i.e. finite but imperfect probability of distinguishability. In the case of perfect indistinguishability, one is left with a state perfectly symmetrized. This wavefunction symmetrization allows one to see indistinguishability as a form of symmetry.

If some particles or states are indistinguishable, there is no observable difference between them, or arising from them. This means that no global variation can be obtained as long as the coherence (under the form of indistinguishability) is not broken by external factors.

For simplicity, consider a bipartite system of two-level components (qubits), leading to a measurement space of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  structure, i.e. a set of global POVM ( $\mathcal{O}$ ) will identify a pair of  $\pm 1$  values. Indistinguishability introduces one or more symmetries in this space across  $\mathbb{Z}_2$  subspaces, that introduces an overall parity-like  $\mathbb{Z}_2$  “charge” linked to which part of the measurement space is occupied. Such symmetry can be a mirror symmetry between a single pair of values, or a full inversion symmetry linking “opposite” pairs, that reduces the measurement space structure to an overall effective  $\mathbb{Z}_2$  space. For example, one may consider the singlet state  $|01\rangle - |10\rangle$  where the two kets are globally indistinguishable, and the global state has overall odd parity, or  $\mathbb{Z}_2$  charge 1. A spin triplet case essentially has an overall  $\mathbb{Z}_2$  “charge” according to parallel or anti-parallel pairings, where the indistinguishable bases are also bound to  $\mathbb{Z}_2$  charge 1 (viz.  $|01\rangle + |10\rangle$ ), with two other (possibly distinguishable) states of charge 0. Therefore, both cases can be treated as bearing a  $\mathbb{Z}_2$  charge, each associated with a different indistinguishable basis set, without loss of generality for the present interest. Furthermore, this  $\mathbb{Z}_2$  charge now encodes the whole non-local state up to an inner phase, and perfect indistinguishability between the two possible states happens when one of the two values of this  $\mathbb{Z}_2$  charge is effectively picked at some point in spacetime. To understand this, note that each  $\mathbb{Z}_2$  charge combines completely indistinguishable states collapsed from the original  $\mathbb{Z}_2 \times \mathbb{Z}_2$  measurement space, and a representation of a state by a linear combination of states bearing different charges would give a finite chance of distinguishability (or restoration of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  space). To prove that such linear combinations cannot coexist with perfect indistinguishability, suppose there is a finite probability of obtaining each pair, labeled  $P_0$  and  $P_1$  according to their  $\mathbb{Z}_2$  charge. They must add to unity, and can be explicitly written as

$$P_0 = P(+1, +1) + P(-1, -1) = 2p, \quad P_1 = P(+1, -1) + P(-1, +1) = 2q. \quad (2)$$

By definition, each pair is indistinguishable, and therefore the probability of obtaining  $P(+1, +1) = P(-1, -1) = p$  and similar for antiparallel measurement. Also, since they add to unity, one obtains that  $p + q = 1/2$ . Therefore,  $P(+1, :) = P(-1, :) = P(:, +1) = P(:, -1) = p + q = 1/2$ . Nevertheless, this breaks the correlation between the two particles completely, rendering them, as well as the constructed state, perfectly distinguishable, i.e. there must be a POVM that distinguishes their state perfectly, with little regard to the

context of the measurement. This contradicts the hypothesis of indistinguishability, proving that such combinations are not allowed if perfect indistinguishability is present and vice-versa. In other words, perfect indistinguishability not only reduces the measurement space but also spontaneously breaks the  $\mathbb{Z}_2$  symmetry of this space, picking one of the two set components.

The first consequence we obtain from our assumption is the raise of Schmidt's rank<sup>20</sup>, i.e. we obtain that indistinguishability can be seen as a source of entanglement (see appendix). While this can be said a known fact (see for instance Eckert et. al<sup>16</sup>) and is not exhaustive, it is a first hint on our claim. For this proof, we use the following lemma.

**Lemma 1** *Given two projectors  $P$  and  $Q$  obeying  $\|P - Q\| < 1$ ,  $\text{rank } P = \text{rank } Q$ .*

A proof of this lemma is presented by Dym<sup>21</sup> and provided in the appendix. Now, to show that Schmidt's rank is raised above 1, we again consider the case of two qubits first. The four projectors on orthonormal bases are written  $\pi_{ij}$  ( $i, j = 0, 1$ ). If we assume at least an instantaneous indistinguishability of two bases related by  $X^{\otimes 2}$  operations, one cannot deal with these projectors isolated, but must combine them into one projector  $\Pi = \alpha\pi_{ij} + \beta\pi_{kl}$ , with  $|\alpha| = |\beta| = 1$ . Without loss of generality, we set both factors to 1 on the rest of this paper. Explicitly, this gives us two possible projectors  $\Pi_0 = \pi_{00} + \pi_{11}$  and  $\Pi_1 = \pi_{01} + \pi_{10}$ , which will be collectively referred to simply as  $\Pi$ . Observe that this new projector is now rank 2, as each  $\pi_{ij}$  is a rank 1 operator projecting on linearly independent orthonormal spaces.

Taking a pure state in Schmidt bases  $|\psi\rangle = \sum_k \lambda_k |\lambda_k^A\rangle |\lambda_k^B\rangle \equiv \sum_k \lambda_k |\lambda_k^{AB}\rangle$ , one can construct a projector onto its Schmidt space as  $S = \sum_k |\lambda_k^{AB}\rangle \langle \lambda_k^{AB}|$ . If such state-space coexist with indistinguishability, i.e. are defined together with the projector  $\Pi$ , it can be shown that if a state occupies bases in  $\Pi$ 's range only, then  $\|S - \Pi\| < 1$ , making both projectors to have the same rank 2; in the general case, for a  $r$ -rank projector  $\Pi$  blending more bases in a higher dimensional Hilbert space, the same result stands and  $S$  is also  $r$ -rank (see appendix). The higher Schmidt rank is a mark of entanglement, which in this case is obtained from indistinguishability, responsible for raising  $\Pi$ 's rank. We may say that the projector  $\Pi$  is an entangler bearing a disentangling relation with its dual (viz.  $\Pi_0 \leftrightarrow \Pi_1$ ), since one's range is another's kernel, a fact that also explains why a single  $\mathbb{Z}_2$  charge must be picked up for perfect indistinguishability, and not a linear combination.

At this point, we may see distinguishability as lack of interaction and lack of correlation, or indistinguishability as a kind of interaction/correlation inducing symmetry. We remember that quantum correlations of two qubits are capped by Tsirelson's bound, which can be written in a form of CHSH inequality,

$$|E_Q(A, B)| = |\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| \leq 2\sqrt{2}. \quad (3)$$

The notation  $\langle . \rangle$  specifically denotes standard quantum correlations, while  $E$ , used in eq. (1), stands for a general case, with  $E_Q$  representing CHSH-like quantum correlations. The indices stand for different bases that do not need to be shared by  $A$  and  $B$ , and can be understood to be mutually independent. To analyze the bounds of these correlations, consider eq. (3), reading all the correlations as a more general "superquantum" possibility. The maximum possible value obtained is 4, which is not allowed in the structure of the quantum mechanical Hilbert space. But what can be said about operators acting on qubits  $A$  and  $B$ ? We can define a set of local orthogonal physical measurement operators  $\{\mathcal{O}_i(r_i)\}$ , and then

$$A_m = \sum_i c_i^{(m)} \mathcal{O}_i(r_A), \quad (4)$$

and similarly replacing  $A$  for  $B$ . The correlations that can be achieved, in principle, have no specific constraint, as shown by Popescu and Rohlich<sup>5</sup>. However, let us consider their setup with CHSH correlation equal to 4, i.e.  $E(A_1, B_1) = E(A_2, B_1) = E(A_1, B_2) = -E(A_2, B_2) = 1$ . Although at a first glance this system seems to respect particle indistinguishability, having symmetric probabilities to measurements on bases  $A, B$ , viz.  $P_{A_1 B_1}(+1, +1) = P_{A_1 B_1}(-1, -1) = P_{A_2 B_1}(+1, +1) = P_{A_2 B_1}(-1, -1) = P_{A_1 B_2}(+1, +1) = P_{A_1 B_2}(-1, -1), P_{A_2 B_2}(-1, +1) = P_{A_2 B_2}(+1, -1)$ , it actually violates indistinguishability and is therefore prohibited if we assume indistinguishability as a fundamental principle to generate correlations, i.e. maximum entanglement bound to maximum indistinguishability.

Let  $E(A_1, B_1) = 1$  under completely indistinguishable bases. On a two-level bipartite system, this assures an even value of the  $\mathbb{Z}_2$  charge linked to a state equivalence. A change

of either of the bases  $A$  or  $B$  must then be equivalent. Therefore, the only difference of changing  $A_1 \rightarrow A_2$  or  $B_1 \rightarrow B_2$  is the difference between  $A_2$  and  $B_2$ , i.e. if  $A_2$  and  $B_2$  are the same measurement basis on different particles (or regions of space), then the expected correlations must still be the same. If  $E(A_1, B_2) = E(A_2, B_1) = 1$ , we are now bound to have the same even parity on both (local) measurements. This does not assure us a global, basis-agnostic parity selection, but leads to an interlocked parity in these specific bases; Measurements in  $A_1$  and  $B_1$  being equivalent create also an equivalence between  $A_2$  and  $B_2$ , and  $E(A_2, B_2)$  should bear the same even  $\mathbb{Z}_2$  charge. Nevertheless, maximum correlation requires  $E(A_2, B_2) = -1$ . Whilst this is a conceivable pattern, it leads to a  $\mathbb{Z}_2$  charge inversion in what are essentially indistinguishable, parity fixed states. Such result could only be conceived if one could always distinguish  $A$  and  $B$ , not only on the time of measurement but when their correlation originated. Hence, taking indistinguishability as a physical principle imposes a limit to the non-local correlations one can get.

This proves that correlations allowed just by no-signaling assumption can be restricted by indistinguishability assumption. The remaining matter becomes: does indistinguishability lead to Tsirelson's bound? We will derive it now from some basic indistinguishability assumptions addressed in this text.

First, consider different bases  $A_1, A_2, B_1, B_2$ , defined according to eq. (4), used to calculate correlations given by CHSH inequality in equation (3). Assuming that correlations are bound by indistinguishability only, this means that basis indistinguishability must play a major role limiting the possible correlations found. Suppose we take an arbitrary set for bases  $A$  and  $B$ . For an eigenstate of indistinguishable basis in the range of  $\Pi$ , the maximum expected correlations measured on it can be written

$$\begin{aligned}
E(A_m, B_n) &= \text{Tr}(\Pi A_m B_n) \\
&= \text{Tr} \left( \Pi \sum_{ij} c_i^{(m)} c_j^{(n)} \mathcal{O}_i(r_A) \mathcal{O}_j(r_B) \right) \\
&= \sum_{ij} c_i^{(m)} c_j^{(n)} \text{Tr}(\Pi \mathcal{O}_i(r_A) \mathcal{O}_j(r_B)) \\
&= \sum_{i \neq j} c_i^{(m)} c_j^{(n)} \text{Tr}(\Pi \mathcal{O}_i(r_A) \mathcal{O}_j(r_B)) + \sum_k c_k^{(m)} c_k^{(n)} \text{Tr}(\Pi \mathcal{O}_k(r_A) \mathcal{O}_k(r_B)). \quad (5)
\end{aligned}$$

Eq. (5) separates the relevant correlation as a sum of orthogonal (1st term) and parallel



(2nd term) measurements. The indistinguishability principle discussed here assures a full symmetry between state components that grants a  $\mathbb{Z}_2$ -symmetry breaking similar to a parity locking. In fact, for Bell pairs like a singlet state  $|s\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ , one can understand an odd parity present, while a Bell pair like  $(|00\rangle + |11\rangle)/\sqrt{2}$  can be seen to have an even parity. Both cases can be seen as examples of a state indistinguishability following definition 1, which is based on the symmetry between  $|0\rangle$  and  $|1\rangle$  states. We may observe that the first term in eq. (5) vanishes, leaving only the second term, by noticing that the parallel case with  $i = j$  generates an operator  $\mathcal{O}_{kk} \equiv \mathcal{O}_k(r_A)\mathcal{O}_k(r_B)$  that has the same range as  $\Pi$ , since  $\mathcal{O}_{kk}$  cannot generate any distinguishable measurement by itself and therefore has no means to go beyond  $\Pi$  range. On the other hand,  $\mathcal{O}_{ij} \equiv \mathcal{O}_i(r_A)\mathcal{O}_j(r_B)$  can be expanded onto the same basis as  $\Pi_0$  and  $\Pi_1$  as

$$\mathcal{O}_{ij} = \sum_{mnpq} \tilde{c}_{mnpq} |mp\rangle\langle nq|, \quad \tilde{c}_{mnpq} = c_{mn}c_{pq}, \quad c_{\alpha\beta} = c_{\beta\alpha}^*, \quad (6)$$

for  $\Pi = \sum_k d_k \pi_k = \sum_k d_k |kk'\rangle\langle kk'|$  ( $k'$  bound to  $k$ ). Hence, the orthogonal term of eq. (5) becomes

$$\begin{aligned} \text{Tr}(\Pi \mathcal{O}_{ij}) &= \sum_{k,i,j} c_i^{(m)} c_j^{(n)} d_k \text{Tr}(\pi_k \mathcal{O}_i(r_A) \mathcal{O}_j(r_B)) \\ &= \sum_{knmpq} d_k \tilde{c}_{mnpq} \text{Tr}(|kk'\rangle\langle kk'| |mp\rangle\langle nq|). \end{aligned} \quad (7)$$

As can be immediately seen, if the basis  $|mp\rangle$  and  $|nq\rangle$  do not both coincide with that in  $\pi_k$  (what may only happen in the parallel term), the above trace vanishes. Eq. (5) reduces to

$$E(A_m, B_n) = \sum_k c_k^{(m)} c_k^{(n)} \text{Tr}(\Pi \mathcal{O}_k(r_A) \mathcal{O}_k(r_B)). \quad (8)$$

The maximum correlation one can expect occurs when  $\text{Tr}(\Pi \mathcal{O}_k(r_A) \mathcal{O}_k(r_B)) = \pm 1$  for every  $k$ , when  $\mathcal{O}_{kk}$  executes linear transformations within  $\Pi$ 's range. Noting that  $c_i^{(m,n)}$  obeys the constraint

$$\sum_i |c_i^{(m)}|^2 = \sum_i |c_i^{(n)}|^2 = 1, \quad (9)$$

one can analyze when eq. (8) takes its maximum value by taking the minimum number of components possible. In the case when only a single component of  $c_i^{(m)}$ 's becomes nonzero for

all  $A_{1,2}$  and  $B_{1,2}$ , eq. (8) reduces to  $E(A_m, B_n) = \pm 1$ , according to the  $\mathbb{Z}_2$  charge present. Since the sign is fixed, CHSH correlation factor becomes 2 identically. To overcome this constraint, one may try to increase the number of nonzero components of observation in either  $A$ ,  $B$ , or both. For single component  $A_1 = \mathcal{O}_1$  and  $B_1 = \mathcal{O}_1$ , and double component  $A_2 = c_1^A \mathcal{O}_1 + c_2^A \mathcal{O}_2$  and  $B_2 = c_1^B \mathcal{O}_1 + c_2^B \mathcal{O}_2$ , we have

$$\begin{aligned} E(A_1, B_1) &= \pm 1, & E(A_1, B_2) &= \pm c_1^B \\ E(A_2, B_1) &= \pm c_1^A, & E(A_2, B_2) &= \pm c_1^A c_1^B \pm c_2^A c_2^B. \end{aligned} \quad (10)$$

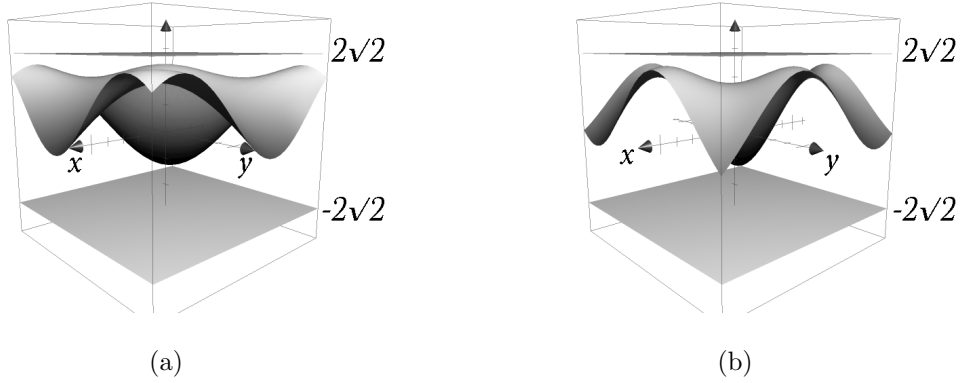


FIG. 1. Correlation function in eq. (11) taken (a) +, (b) -. Given the normalization condition,  $c_1^{A(B)}$  is rewritten as  $\sin x$  ( $y$ ) and  $c_2^{A(B)}$  as  $\cos x$  ( $y$ ). The upper and lower plan indicate  $z = \pm 2\sqrt{2}$ , and axes range from  $-\pi$  to  $\pi$ .

Again, the first component of both  $A_2$  and  $B_2$  will be forced to be the same as  $A_1$  and  $B_1$ , assuring the same sign for the first three correlations, as well as the  $c_1 c_1$  term which will be subtracted from the total correlation, giving us a correlation of the shape

$$E = 1 + c_1^A + c_1^B - c_1^A c_1^B \pm c_2^A c_2^B, \quad (11)$$

not allowing a maximum correlation in such measurements as can be confirmed by plotting the graph of this function (fig. 1). We then take both  $A$  to be single component, and both  $B$  to be two-component. Let  $A_1 = \mathcal{O}_1$ ,  $A_2 = \mathcal{O}_2$ ,  $B_1 = c_1^{(1)} \mathcal{O}_1 + c_2^{(1)} \mathcal{O}_2$ , and  $B_2 = c_1^{(2)} \mathcal{O}_1 + c_2^{(2)} \mathcal{O}_2$ . Note that as  $A$  has a single component and our state symmetry require the same components between  $A$  and  $B$ , only  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are present. Consequently,

$$\begin{aligned}
E(A_1, B_1) &= \pm c_1^{(1)}, & E(A_1, B_2) &= \pm c_1^{(2)} \\
E(A_2, B_1) &= \pm c_2^{(1)}, & E(A_2, B_2) &= \pm c_2^{(2)}.
\end{aligned} \tag{12}$$

Again, signs are bound to state symmetry. However, since each coefficient  $c$  can assume any complex value, we may absorb the signs into the coefficients and write CHSH correlation factor simply as a sum

$$E = c_1^{(1)} + c_2^{(1)} + c_1^{(2)} - c_2^{(2)}, \tag{13}$$

choosing the sign of each coefficient to achieve this expression. With the normalization constraint of eq. (9), one can immediately calculate the maximum value for this correlation to be  $2\sqrt{2}$ , the so called Tsirelson's bound. Also, notice that the operators  $A_i B_j$  created are a linear combination of one operator with the same range as  $\Pi$  ( $\mathcal{O}_{kk}$ ) and one whose range is  $\Pi$ 's kernel ( $\mathcal{O}_{ij}$ ). This interplay between POVMs that take a state within a space of indistinguishable bases ( $\Pi$ 's range, also Schmidt space) to itself and to its dual space ( $\Pi$ 's kernel), creates entanglement witnesses of maximum correlations.

If we further increase the number of components, the individual size of each factor  $c$  would become smaller, leading only to smaller correlations. For instance, increasing only  $A$  or  $B$  components would only reduce the size of each term in eq. (13). Increasing the amount of components of both would introduce more terms of second order to eq. (13), essentially reducing the possible total sum absolute result. In other words, more projections towards  $\Pi$ 's range or kernel would appear, invariably reducing the total possible correlation.

In conclusion, we have introduced indistinguishability as a symmetry in the measurement space, and showed that assuming indistinguishability as a fundamental principle, together with non-locality and no-signaling, one can explain quantum theory bounds to CHSH inequality, explaining why previous work have surpassed Tsirelson's bound and its relation to indistinguishability. Taken as a fundamental principle, indistinguishability may help us to understand the quantum world as not only a probabilistic plaza, but really an equality venue for minute components that cannot be discerned from a macroscopic scale in size and component amounts. This principle may also shed new light on paradoxes like Schrödinger's cat, which could only be possible if the states  $|alive\rangle$  and  $|dead\rangle$  (microscopically composed by very largely entangled qubit state, and not a single qubit superposition) were indistinguishable at some point in spacetime, what is likely not to be the case.

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- <sup>18</sup> For all the matters at this point, only identical particles are being considered. Nevertheless, the same principle can be extended to non-identical particles. Consider a Hydrogen atom, for instance. It consists of two distinct particles with opposite charge; once they form a bound state, can they really be distinguished without isolating each other? If they form a spin singlet state, are their spins really distinguishable, or the singlet pairing process is a product of the complete loss of spin distinguishability? Can their charge be identified and assigned without separating them far apart (ionizing the atom) and cutting the interaction present? Under these considerations, the same concept of indistinguishability can be applied to non-identical particles interacting in a region of spacetime, until they are far apart. Therefore, no explicit mention to particle identity is made in this definition.

- <sup>19</sup> For now, we shall only look in the 3+1D case, where spin-statistics theorem applies, and no complications from anyonic (Abelian or non-Abelian) cases arise. This might be considered in future work.
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## Appendix A: Tsirelson's Bound

A brief proof of Tsirelson's bound is given following Peres<sup>1</sup>. We write

$$C = \alpha\beta + \gamma\beta + \alpha\delta - \gamma\delta, \quad (\text{A1})$$

assuming they either commute or anti-commute, i.e.  $[\alpha, \beta]_{\pm} = [\beta, \gamma]_{\pm} = [\gamma, \delta]_{\pm} = [\delta, \alpha]_{\pm} = 0$ . Hence, one can write

$$C^2 = \mp 4 + [\alpha, \gamma]_- [\beta, \delta]_-, \quad (\text{A2})$$

$$\|C^2\| \leq 4 + \|[\alpha, \gamma]\| \|[\beta, \delta]\|$$

$$\|C^2\| \leq 4 + 4\|\alpha\gamma\| \|\beta\delta\|$$

$$\therefore \|C\| \leq 2\sqrt{2}. \quad (\text{A3})$$

This can be understood by noticing that when squaring  $C$ , terms like  $\alpha\beta\gamma\beta$  cancel out with terms like  $-\alpha\delta\gamma\delta$ , whether these operators commute or anti-commute, leaving only a  $\alpha\beta\gamma\delta$  term.

## Appendix B: Proof of lemma 1

If  $\|P - Q\| < 1$ , clearly  $I_n - (P - Q)$  is full rank and invertible. Hence:

$$\begin{aligned} \text{rank } P &= \text{rank } P(I_n - (P - Q)) = \text{rank } PQ \\ &\leq \text{rank } Q. \end{aligned} \quad (\text{B1})$$

The second equality comes from projectors' idempotency ( $P^2 = P$ ). As the same argument can be made exchanging  $P$  and  $Q$ , they must have the same rank. QED.

### Appendix C: Schmidt decomposition

Given a bipartite quantum system, it can in general be represented as

$$|\Psi\rangle = \sum_{ij} c_{ij} |i\rangle |j\rangle. \quad (\text{C1})$$

By performing a singular value decomposition of the coefficient matrix, one obtains

$$\begin{aligned} |\Psi\rangle &= \sum_{ijk} v_{ik} \lambda_k u_{kj} |i\rangle |j\rangle \\ &= \sum_k \lambda_k |\lambda_k^A\rangle |\lambda_k^B\rangle, \end{aligned} \quad (\text{C2})$$

where  $\lambda_k$  are the singular values, and the correlated kets in the second line are obtained by applying  $v_{ik}$  ( $u_{kj}$ ) to  $|i\rangle$  ( $|j\rangle$ ). These are called Schmidt bases, and the rank of the matrix  $\Lambda = (\lambda_k)$ , i.e. the number of singular values, is called Schmidt rank. A Schmidt rank equal to 1 implies that the quantum state is separable into the product of two independent states, and is therefore unentangled by definition. For this reason, Schmidt rank can be used as a measure of entangled. For more, see for instance Nielsen and Chuang<sup>20</sup>.

### Appendix D: Proof of Schmidt's rank raise

For the Schmidt space projector  $S = \sum_k |\lambda_k\rangle \langle \lambda_k|$  defined in the main text, one can revert it to the same basis as an  $r$ -rank ( $r > 1$ ) projector  $\Pi$ :

$$\begin{aligned} S &= \sum_k |\lambda_k\rangle \langle \lambda_k| \\ &= \sum_{ijk} v_{ik} u_{jk} |ij\rangle \langle ij| v_{ik}^* u_{jk}^* = \sum_{ijk} |v_{ik}|^2 |u_{jk}|^2 |ij\rangle \langle ij|. \end{aligned} \quad (\text{D1})$$

If we suppose that only one Schmidt basis exist, the summation over  $k$  is trivial and we are left with  $|v_i| = |u_j| = 1$  in order to have a norm 1 projector. Then, we obtain

$$S - \Pi = \sum_{ij \notin \Pi} |ij\rangle \langle ij|. \quad (\text{D2})$$

If  $S$  contains bases not in  $\Pi$ , we are still left with a norm 1 operator and a non-entangled state is possible, as assumed. However, if only the bases in  $\Pi$  are present, this should be a norm 0 operator, and according to lemma 1, should bear the same rank as  $\Pi$ . Indeed, if one does not ignore the summation over  $k$  and assumes  $|v_{ik}|, |u_{jk}| < 1$ , a finite component of this difference remains, but still one has  $\|S - \Pi\| < 1$ , reassuring that  $k = r > 1$ . Hence, for a state within  $\Pi$ 's range, Schmidt rank is greater than 1.